Question Proposal

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1 The Problem

The problem may be given in different styles. Notably, the full unrestrained version of the problem would be:

Prove **using geometry** that $\sqrt{x^{2n}} = x^n,$ given $x > 0$.

However, for ease, the question could instead be given as two parts:

Prove **using geometry** that given $x > 0$:

a) $\sqrt{x^4} = x^2$.

b) Hence, or otherwise, that $\sqrt{x^{2n}} = x^n$.

This problem looks deceptively simple, especially for higher-level mathematics. The formulae which we are trying to prove can be deducted using the basic laws of algebra. The twist is that we are asked to derive them *geometrically*, which in my opinion urges the person being asked the question to think of the problem differently.

Additionally, the solution shared in Section 2 is in my eyes quite elegant, with a climax when everything begins cancelling out in the ever-chaotic Heron's formula.

2 Solution

To start, we will prove the easier fact that for $x > 0$, $\sqrt{x^4} = x^2$ geometrically. It is worth mentioning that the question specifically asks for geometry to be used *because* of the restriction $x > 0$ (lengths in geometry also abide by this restriction) - it is no coincidence.

2.1 Proving $\sqrt{x^4} = x^2$

Let us construct an isosceles triangle $\triangle ABC$ such that the length AB is x, as shown in Figure 1.

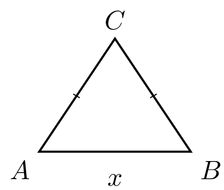

Figure 1: $\triangle ABC$

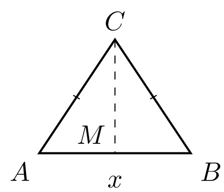

Figure 2: $\triangle ABC$ with the altitude MC, shown by the dotted line.

Let M be the midpoint of AB. We now construct the altitude MC. This is shown in Figure 2. Let the area of the triangle be x^2. Hence, because the length is x, the length of the altitude MC is $2x$.[1] Now, we can find the length BC through the Pythagorean theorem, as it is the hypotenuse of $\triangle BMC$. MC is $2x$ and MB is $\frac{x}{2}$, therefore, BC is $\sqrt{(\frac{x}{2})^2 + (2x)^2}$. Because it is an Isosceles triangle, we can deduce that $AC = BC = \sqrt{(\frac{x}{2})^2 + (2x)^2}$ also. For simplicity, we will let $AC = BC = \sqrt{(\frac{x}{2})^2 + (2x)^2} = h$, a variable which we can use later.

Here is where the real magic begins. Now that we have all the lengths of $\triangle ABC$, we can find its area also using Heron's formula. We calculate the semi-perimeter to be $s = \frac{x+2h}{2}$. Hence,

$$\begin{aligned}
\text{Area}(\triangle ABC) &= \sqrt{s(s-x)(s-h)(s-h)} \\
&= \sqrt{\left(\frac{x+2h}{2}\right)\left(\frac{x+2h}{2} - x\right)\left(\frac{x+2h}{2} - h\right)\left(\frac{x+2h}{2} - h\right)} \\
&= \sqrt{\left(\frac{x+2h}{2}\right)\left(\frac{2h-x}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)} \\
&= \sqrt{\frac{(x+2h)(2h-x)(x^2)}{16}} \\
&= \sqrt{\frac{(2hx - x^2 + 4h^2 - 2hx)(x^2)}{16}} \\
&= \sqrt{\frac{(4h^2 - x^2)(x^2)}{16}}.
\end{aligned}$$

[1]So that the area of $\triangle ABC$ is $\frac{1}{2} \times 2x \times x = x^2$.

Substituting $h = \sqrt{(\frac{x}{2})^2 + (2x)^2} \implies h^2 = \frac{x^2}{4} + 4x^2$, we get

$$\begin{aligned}
\sqrt{\frac{(4h^2 - x^2)(x^2)}{16}} &= \sqrt{\frac{\left(4\left(\frac{x^2}{4} + 4x^2\right) - x^2\right)(x^2)}{16}} \\
&= \sqrt{\frac{(x^2 + 16x^2 - x^2)(x^2)}{16}} \\
&= \sqrt{\frac{(16x^2)(x^2)}{16}} \\
&= \sqrt{\frac{16x^4}{16}} \\
&= \sqrt{x^4}.
\end{aligned}$$

Because we found earlier that the area of $\triangle ABC$ is equal to x^2, we have

$$\begin{aligned}
\text{Area}(\triangle ABC) &= \text{Area}(\triangle ABC) \\
\implies x^2 &= \sqrt{x^4}.
\end{aligned}$$

Thus, by LHS-RHS, we have proven that $x^2 = \sqrt{x^4}$. □

2.2 Proving $\sqrt{x^{2n}} = x^n$

In the interest of keeping this document short, the proof that $\sqrt{x^{2n}} = x^n$ will not be included. To prove it, you would use the same method as shown in Section 2.1, but where instead of letting AB be x and MC be $2x$, you let AB be $x^{\frac{1}{2}n}$ and let MC be $2x^{\frac{1}{2}n}$. Thus, the area of $\triangle ABC$ would be $\frac{1}{2} \times x^{\frac{1}{2}n} \times 2x^{\frac{1}{2}n} = x^n$, yielding the required result. For assurance, the numbers have been crunched and indeed we do get $\sqrt{x^{2n}} = x^n$. [1]

References

[1] Calculated by Wolfram Alpha at `https://www.wolframalpha.com/input?i2d=true&i=A%3D%5C%2840%29%5C%2840%29Divide%5B1%2C2%5D%5C%2841%29%5C%2840%29Power%5Bx%2CDivide%5B1%2C2%5Dn%5D%5C%2841%29%5C%2840%292Power%5Bx%2CDivide%5B1%2C2%5Dn%5D%5C%2841%29%5C%2841%29%5C%2844%29+H%3D%5C%2840%29Sqrt%5BPower%5B%5C%2840%29Divide%5BPower%5Bx%2CDivide%5B1%2C2%5Dn%5D%2C2%5D%5C%2841%29%2C2%5D%2BPower%5B%5C%2840%292Power%5Bx%2CDivide%5B1%2C2%5Dn%5D%5C%2841%29%2C2%5D%5D%5C%2841%29%5C%2844%29+s%3DDivide%5BPower%5Bx%2CDivide%5B1%2C2%5Dn%5D%2BH%2BH%2C2%5D%5C%2844%29+findSqrt%5Bs%5C%2840%29s-Power%5Bx%2CDivide%5B1%2C2%5Dn%5D%5C%2841%29%5C%2840%29s-H%5C%2841%29%5C%2840%29s-H%5C%2841%29%5D`.